

Axially symmetric solutions in $f(R)$ -gravity

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Axially symmetric solutions for $f(R)$ -gravity can be derived starting from exact spherically symmetric solutions achieved by Noether symmetries. The method takes advantage of a complex coordinate transformation previously developed by Newman and Janis in General Relativity. An example is worked out to show the general validity of the approach. The physical properties of the solution are also considered.

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I. INTRODUCTION

The issue to extend General Relativity (GR) to alternative theories of gravity has recently become dramatically urgent due to the missing matter problem at all astrophysical scales and the accelerating behavior of cosmic fluid, detected by SuperNovae Ia used as standard candles. Up to now, no final answer on new particles has been given at fundamental level so Dark Energy and Dark Matter constitute a puzzle to be solved in order to achieve a self-consistent picture of the observed Universe. $f(R)$ -gravity, where $f(R)$ is a generic function of the Ricci scalar R , comes into the game as a straightforward extension of GR where further geometrical degrees of freedom are considered instead of searching for new material ingredients [1]. From an epistemological point of view, the action of gravity is not selected *a priori*, but it could be "reconstructed", in principle, by matching consistently the observations [2–4]. This approach can be adopted considering any function of the curvature invariants as $R_{\mu\nu}R^{\mu\nu}$, $R\Box R$ and so on.

From a genuine mathematical point of view, alternative theories of gravity pose the problem to recover or extend the well-established results of GR as the initial value problem [5], the stability of solutions and, in particular, the issue of finding out new solutions. As it is well known, beside cosmological solutions, spherically and axially symmetric solutions play a fundamental role in several astrophysical problems ranging from black holes to active galactic nuclei. Alternative gravities, to be consistent with results of GR, should comprise solutions like Schwarzschild and Kerr ones but present, in general, new solutions that could be physically interesting. Due to this reason, methods to find out exact and approximate solutions are particularly relevant in order to check if observations can be framed in Extended Theories of Gravity [6].

Recently, the interest in spherically symmetric solutions of $f(R)$ -gravity is growing up. In [7], solutions in vacuum have been found considering relations among functions that define the spherical metric or imposing a constant Ricci curvature scalar. The authors have reconstructed the form of some $f(R)$ -models, discussing their physical relevance. In [8], the same authors have discussed static spherically symmetric solutions, in presence of perfect fluid matter, adopting the metric formalism. They have shown that a given matter distribution is not capable of globally determining the functional form of $f(R)$. Others authors have discussed in details the spherical symmetry of $f(R)$ -gravity considering also the relations with the weak field limit. Exact solutions are obtained for constant Ricci curvature scalar and for Ricci scalar depending on the radial coordinate. In particular, it can be considered how to obtain results consistent with GR assuming the well-known post-Newtonian and post-Minkowskian limits as consistency checks. [9].

In this paper, we want to seek for a general method to find out axially symmetric solutions by performing a complex coordinate transformation on the spherical metrics. Since the discovery of the Kerr solution [10], many attempts have been made to find a physically reasonable interior matter distribution that may be considered as its source. For a review on these approaches see [11, 12]. Though much progress has been made, results have been generally disappointing. As far as we know, nobody has obtained a physically satisfactory interior solution. This seems surprising given the success of matching internal spherically symmetric solutions to the Schwarzschild metric. The problem is that the loss of a degree of symmetry makes the derivation of analytic results much more difficult. Severe restrictions are placed on the interior metric by maintaining that it must be joined smoothly to the external axially symmetric metric. Further restrictions are placed on the interior solutions to ensure that they correspond to physical objects.

Furthermore since the axially symmetric metric has no radiation field associated with it, its source should be also non-radiating. This places even further constraints on the structure of the interior solution [13]. Given the strenuous nature of these limiting conditions, it is not surprising to learn that no satisfactory solution to the problem of finding

sources for the Kerr metric has been obtained. In general, the failure is due to internal structures whose physical properties are unknown. This shortcoming makes hard to find consistent boundary conditions.

Newman and Janis showed that it is possible to obtain an axially symmetric solution (like the Kerr metric) by making an elementary complex transformation on the Schwarzschild solution [15]. This same method has been used to obtain a new stationary and axially symmetric solution known as the Kerr-Newman metric [16]. The Kerr-Newman space-time is associated to the exterior geometry of a rotating massive and charged black-hole. For a review on the Newman-Janis method to obtain both the Kerr and Kerr-Newman metrics see [17].

By means of very elegant mathematical arguments, Schiffer et al. [18] have given a rigorous proof to show how the Kerr metric can be derived starting from a complex transformation on the Schwarzschild solution. We will not go into the details of this demonstration, but point out that the proof relies on two main assumptions. The first is that the metric belongs to the same algebraic class of the Kerr-Newman solution, namely the Kerr-Schild class [19]. The second assumption is that the metric corresponds to an empty solution of the Einstein field equations. In the case we are going to study, these assumptions are not considered and hence the proof in [18] is not applicable. It is clear, by the generation of the Kerr-Newman metric, that all the components of the stress-energy tensor need to be non-zero for the Newman-Janis method to be successful. In fact, Gürses and Gürsey, in 1975 [20], showed that if a metric can be written in the Kerr-Schild form, then a complex transformation “is allowed in General Relativity.” In this paper, we will show that such a transformation can be extended to $f(R)$ -gravity.

The outline of this paper is as follows. In the Sec.II, we introduce the $f(R)$ -gravity action, the field equations and give some general remarks on spherical symmetry. In Sec. III, we give a summary of the Noether Symmetry Approach [6] and find some spherically symmetric exact solutions for $f(R)$ -gravity. In Sec. IV, we review the Newman-Janis method to obtain axially symmetric solutions starting from spherically symmetric ones. The resulting metric is written in terms of two arbitrary functions. A further suitable coordinate transformation allows to write the metric in the so called Boyer-Lindquist coordinates. Such a transformation makes the physical interpretation much clearer and reduces the amount of algebra required to calculate the metric properties. In Sec.V, the Newman-Janis method is applied to the spherically symmetric exact solution, previously derived by the Noether Symmetry, and an axially symmetric exact solution is obtained. This result shows that the Newman-Janis method works also in $f(R)$ -gravity. A physical application of the result is discussed in Sec VI. Discussion and concluding remarks are drawn in Sec. VII.

II. SPHERICAL SYMMETRY IN $f(R)$ -GRAVITY

Let us consider an analytic function $f(R)$ of the Ricci scalar R in four dimensions. The variational principle for this action is:

$$\delta \int d^4x \sqrt{-g} \left[f(R) + \mathcal{X} \mathcal{L}_m \right] = 0 \quad (1)$$

where $\mathcal{X} = \frac{8\pi G}{c^4}$, \mathcal{L}_m is the standard matter Lagrangian and g is the determinant of the metric¹.

By varying with respect to the metric, we obtain the field equations²

$$\begin{cases} H_{\mu\nu} = f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - f'(R)_{;\mu\nu} + g_{\mu\nu}\square f(R) = \mathcal{X}T_{\mu\nu} \\ H = g^{\rho\sigma}H_{\rho\sigma} = 3\square f(R) + f'(R)R - 2f(R) = \mathcal{X}T \end{cases} \quad (2)$$

where $T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$ is the energy-momentum tensor of standard fluid matter and the second equation is the trace. The most general spherically symmetric solution can be written as follows:

¹ We are adopting the convention $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ for the Ricci tensor and $R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \dots$, for the Riemann tensor. Connections are Levi-Civita :

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\rho}(g_{\alpha\rho,\beta} + g_{\beta\rho,\alpha} - g_{\alpha\beta,\rho}).$$

² It is possible to take into account also the Palatini approach in which the metric g and the connection Γ are considered independent variables (see for example [21]). Here we will consider the Levi-Civita connection and will use the metric approach. See [3, 22] for a detailed comparison between the two pictures.

$$ds^2 = m_1(t', r') dt'^2 + m_2(t', r') dr'^2 + m_3(t', r') dt' dr' + m_4(t', r') d\Omega, \quad (3)$$

where m_i are functions of the radius r' and of the time t' . $d\Omega$ is the solid angle. We can consider a coordinate transformation that maps the metric (3) in a new one where the off-diagonal term vanishes and $m_4(t', r') = -r^2$, that is³:

$$ds^2 = g_{tt}(t, r) dt^2 - g_{rr}(t, r) dr^2 - r^2 d\Omega. \quad (4)$$

This expression can be considered, without loss of generality, as the most general definition of a spherically symmetric metric compatible with a pseudo-Riemannian manifold without torsion. Actually, by inserting this metric into the field Eqs. (2), one obtains:

$$\begin{cases} f'(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} + \mathcal{H}_{\mu\nu} = \mathcal{X}T_{\mu\nu} \\ f'(R)R - 2f(R) + \mathcal{H} = \mathcal{X}T \end{cases} \quad (5)$$

where the two quantities $\mathcal{H}_{\mu\nu}$ and \mathcal{H} read:

$$\begin{aligned} \mathcal{H}_{\mu\nu} = -f''(R) \left\{ R_{,\mu\nu} - \Gamma_{\mu\nu}^t R_{,t} - \Gamma_{\mu\nu}^r R_{,r} - g_{\mu\nu} \left[\left(g^{tt}_{,t} + g^{tt} (\ln \sqrt{-g})_{,t} \right) R_{,t} + \left(g^{rr}_{,r} + g^{rr} (\ln \sqrt{-g})_{,r} \right) R_{,r} + \right. \right. \\ \left. \left. + g^{tt} R_{,tt} + g^{rr} R_{,rr} \right] \right\} - f'''(R) \left[R_{,\mu} R_{,\nu} - g_{\mu\nu} \left(g^{tt} R_{,t}^2 + g^{rr} R_{,r}^2 \right) \right] \end{aligned} \quad (6)$$

$$\begin{aligned} \mathcal{H} = g^{\sigma\tau} \mathcal{H}_{\sigma\tau} = 3f''(R) \left[\left(g^{tt}_{,t} + g^{tt} (\ln \sqrt{-g})_{,t} \right) R_{,t} + \left(g^{rr}_{,r} + g^{rr} (\ln \sqrt{-g})_{,r} \right) R_{,r} + g^{tt} R_{,tt} + g^{rr} R_{,rr} \right] + \\ + 3f'''(R) \left[g^{tt} R_{,t}^2 + g^{rr} R_{,r}^2 \right]. \end{aligned} \quad (7)$$

Our task is now to find out exact spherically symmetric solutions.

In the case of time-independent metric, i.e., $g_{tt} = a(r)$ and $g_{rr} = b(r)$, the Ricci scalar can be recast as a Bernoulli equation of index two with respect to the metric potential $b(r)$ (see [9] for details):

$$b'(r) + \left\{ \frac{r^2 a'(r)^2 - 4a(r)^2 - 2ra(r)[2a(r)' + ra(r)'']}{ra(r)[4a(r) + ra'(r)]} \right\} b(r) + \left\{ \frac{2a(r)}{r} \left[\frac{2 + r^2 R(r)}{4a(r) + ra'(r)} \right] \right\} b(r)^2 = 0. \quad (8)$$

where $R = R(r)$ is the Ricci scalar. A general solution of (8) is:

$$b(r) = \frac{\exp[-\int dr h(r)]}{K + \int dr l(r) \exp[-\int dr h(r)]}, \quad (9)$$

where K is an integration constant while $h(r)$ and $l(r)$ are two functions that, according to Eq.(8), define the coefficients of the quadratic and the linear term with respect to $b(r)$ [23]. We can fix $l(r) = 0$; this choice allows to find out solutions with a Ricci scalar scaling as $-\frac{2}{r^2}$ in term of the radial coordinate. On the other hand, it is not possible to have $h(r) = 0$ since, otherwise, we get imaginary solutions. A particular consideration deserves the limit $r \rightarrow \infty$. In order to achieve a gravitational potential $b(r)$ with the correct Minkowski limit, both $h(r)$ and $l(r)$ have to go to zero at infinity, provided that the quantity $r^2 R(r)$ turns out to be constant: this result implies $b'(r) = 0$, and, finally, also the metric potential $b(r)$ has a correct Minkowski limit.

³ This condition allows to obtain the standard definition of the circumference with radius r .

In general, if we ask for the asymptotic flatness of the metric as a feature of the theory, the Ricci scalar has to evolve to infinity as r^{-n} with $n \geq 2$. Formally, it has to be:

$$\lim_{r \rightarrow \infty} r^2 R(r) = r^{-n}, \quad (10)$$

with $n \in \mathbb{N}$. Any other behavior of the Ricci scalar could affect the requirement to achieve a correct asymptotic flatness.

The case of constant curvature is equivalent to GR with a cosmological constant and the solution is time independent. This result is well known (see, for example, [24]) but we report, for the sake of completeness, some considerations related with it in order to deal with more general cases where a radial dependence for the Ricci scalar is supposed. If the scalar curvature is constant ($R = R_0$), field Eqs.(5), being $\mathcal{H}_{\mu\nu} = 0$, reduce to:

$$\begin{cases} f'_0 R_{\mu\nu} - \frac{1}{2} f_0 g_{\mu\nu} = \mathcal{X} T_{\mu\nu} \\ f'_0 R_0 - 2f_0 = \mathcal{X} T \end{cases} \quad (11)$$

where $f(R_0) = f_0$, $f'(R_0) = f'_0$. A general solution, when one considers a stress-energy tensor of perfect-fluid $T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$, is

$$ds^2 = \left(1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - \lambda}{3} r^2 \right) dt^2 - \frac{dr^2}{1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - \lambda}{3} r^2} - r^2 d\Omega. \quad (12)$$

when $p = -\rho$, $\lambda = -\frac{f_0}{2f'_0}$ and $q^{-1} = f'_0$. This result means that any $f(R)$ -model, in the case of constant curvature, exhibits solutions with de Sitter-like behavior. This is one of the reasons why the dark energy issue can be addressed using these theories [1].

If $f(R)$ is analytic, it is possible to write the series:

$$f(R) = \Lambda + \Psi_0 R + \Psi(R), \quad (13)$$

where Ψ_0 is a coupling constant, Λ plays the role of the cosmological constant and $\Psi(R)$ is a generic analytic function of R satisfying the condition

$$\lim_{R \rightarrow 0} R^{-2} \Psi(R) = \Psi_1, \quad (14)$$

where Ψ_1 is a constant. If we neglect the cosmological constant Λ and Ψ_0 is set to zero, we obtain a new class of theories which, in the limit $R \rightarrow 0$, does not reproduce GR (from Eq.(14), we have $\lim_{R \rightarrow 0} f(R) \sim R^2$). In such a case, analyzing the whole set of Eqs.(11), one can observe that both zero and constant $\neq 0$ curvature solutions are possible. In particular, if $R = R_0 = 0$ field equations are solved for any form of gravitational potential entering the spherically symmetric background, provided that the Bernoulli Eq. (8), relating these functions, is fulfilled for the particular case $R(r) = 0$. The solutions are thus defined by the relation

$$b(r) = \frac{\exp[-\int dr h(r)]}{K + 4 \int \frac{dr a(r) \exp[-\int dr h(r)]}{r[a(r) + r a'(r)]}}, \quad (15)$$

being $g_{tt}(t, r) = b(r)$ from Eq.(4). In [9], some examples of $f(R)$ -models admitting solutions with constant $\neq 0$ or null scalar curvature are discussed.

III. THE NOETHER SYMMETRY APPROACH AND THE SPHERICAL SYMMETRY

Besides spherically symmetric solutions with constant curvature scalar, also solutions with the Ricci scalar depending on radial coordinate r are possible in $f(R)$ -gravity [9]. Furthermore, spherically symmetric solutions can be achieved starting from a point-like $f(R)$ -Lagrangian [6]. Such a Lagrangian can be obtained by imposing the spherical symmetry directly in the action (1). As a consequence, the infinite number of degrees of freedom of the original field theory

will be reduced to a finite number. The technique is based on the choice of a suitable Lagrange multiplier defined by assuming the Ricci scalar, argument of the function $f(R)$ in spherical symmetry.

Starting from the above considerations, a static spherically symmetric metric can be expressed as

$$ds^2 = A(r)dt^2 - B(r)dr^2 - M(r)d\Omega, \quad (16)$$

and then the point-like $f(R)$ Lagrangian⁴ is

$$\mathcal{L} = -\frac{A^{1/2}f'}{2MB^{1/2}}M'^2 - \frac{f'}{A^{1/2}B^{1/2}}A'M' - \frac{Mf''}{A^{1/2}B^{1/2}}A'R' - \frac{2A^{1/2}f''}{B^{1/2}}R'M' - A^{1/2}B^{1/2}[(2+MR)f' - Mf], \quad (17)$$

which is canonical since only the configuration variables and their first order derivatives with respect to the radial coordinate r are present. Details of calculations are in [6]. Eq. (17) can be recast in a more compact form introducing the matrix representation :

$$\mathcal{L} = \underline{q}'^t \hat{T} \underline{q}' + V \quad (18)$$

where $\underline{q} = (A, B, M, R)$ and $\underline{q}' = (A', B', M', R')$ are the generalized positions and velocities associated to \mathcal{L} . It is easy to check the complete analogy between the field equation approach and point-like Lagrangian approach [6].

In order to find out solutions for the Lagrangian (17), we can search for symmetries related to cyclic variables and then reduce dynamics. This approach allows, in principle, to select $f(R)$ -gravity models compatible with spherical symmetry. As a general remark, the Noether Theorem states that conserved quantities are related to the existence of cyclic variables into dynamics [25–27].

It is worth noticing that the Hessian determinant of Eq. (17), $\left\| \frac{\partial^2 \mathcal{L}}{\partial q'_i \partial q'_j} \right\|$, is zero. This result clearly depends on the absence of the generalized velocity B' into the point-like Lagrangian. As matter of fact, using a point-like Lagrangian approach implies that the metric variable B does not contribute to dynamics, but the equation of motion for B has to be considered as a further constraint equation. Then the Lagrangian (17) has three degrees of freedom and not four, as one should expect *a priori*.

Now, since the equation of motion describing the evolution of the metric potential B does not depend on its derivative, it can be explicitly solved in term of B as a function of the other coordinates :

$$B = \frac{2M^2 f'' A' R' + 2M f' A' M' + 4AM f'' M' R' + A f' M'^2}{2AM[(2+MR)f' - Mf]}. \quad (19)$$

By inserting Eq.(19) into the Lagrangian (17), we obtain a non-vanishing Hessian matrix removing the singular dynamics. The new Lagrangian reads⁵

$$\mathcal{L}^* = \mathbf{L}^{1/2} \quad (20)$$

with

$$\mathbf{L} = \underline{q}'^t \hat{\mathbf{L}} \underline{q}' = \frac{[(2+MR)f' - fM]}{M} [2M^2 f'' A' R' + 2MM'(f' A' + 2A f'' R') + A f' M'^2].$$

If one assumes the spherical symmetry, the role of the *affine parameter* is played by the coordinate radius r . In this case, the configuration space is given by $\mathcal{Q} = \{A, M, R\}$ and the tangent space by $\mathcal{TQ} = \{A, A', M, M', R, R'\}$. On the other hand, according to the Noether Theorem, the existence of a symmetry for dynamics described by Lagrangian (17) implies a constant of motion. Let us apply the Lie derivative to the (17), we have⁶ :

$$L_{\mathbf{X}} \mathbf{L} = \underline{\alpha} \cdot \nabla_q \mathbf{L} + \underline{\alpha}' \cdot \nabla_{q'} \mathbf{L} = \underline{q}'^t \left[\underline{\alpha} \cdot \nabla_q \hat{\mathbf{L}} + 2 \left(\nabla_q \alpha \right)^t \hat{\mathbf{L}} \right] \underline{q}', \quad (21)$$

⁴ Obviously, the above choices are recovered for $A(r) = a(r)$, $B(r) = b(r)$, and $M(r) = r^2$. Here we deal with A, B, M as a set of coordinates in a configuration space.

⁵ Lowering the dimension of configuration space through the substitution (19) does not affect the dynamics since B is a non-evolving quantity. In fact, inserting Eq. (19) into the dynamical equations given by (17), they coincide with those derived by (17).

⁶ From now on, \underline{q} indicates the vector $\{A, M, R\}$.

that vanishes if the functions $\underline{\alpha}$ satisfy the following system

$$\underline{\alpha} \cdot \nabla_q \hat{\mathbf{L}} + 2(\nabla_q \underline{\alpha})^t \hat{\mathbf{L}} = 0 \longrightarrow \alpha_i \frac{\partial \hat{\mathbf{L}}_{km}}{\partial q_i} + 2 \frac{\partial \alpha_i}{\partial q_k} \hat{\mathbf{L}}_{im} = 0. \quad (22)$$

Solving the system (22) means to find out the functions α_i which assign the Noether vector [25, 26]. However the system (22) implicitly depends on the form of $f(R)$ and then, by solving it, we get also $f(R)$ -models compatible with spherical symmetry. On the other hand, by choosing the $f(R)$ -form, we can explicitly solve (22). As an example, one finds that the system (22) is satisfied if we choose

$$f(R) = f_0 R^s, \quad \underline{\alpha} = (\alpha_1, \alpha_2, \alpha_3) = \left((3-2s)kA, -kM, kR \right), \quad (23)$$

with s a real number, k an integration constant and f_0 a dimensional coupling constant⁷. This means that, for any $f(R) = R^s$, exists, at least, a Noether symmetry and a related constant of motion Σ_0 :

$$\Sigma_0 = \underline{\alpha} \cdot \nabla_q \mathbf{L} = 2skMR^{2s-3}[2s + (s-1)MR][(s-2)RA' - (2s^2 - 3s + 1)AR'].$$

A physical interpretation of Σ_0 is possible if one gives an interpretation of this quantity in GR, that means for $f(R) = R$ and $s = 1$. In other words, the above procedure has to be applied to the Lagrangian of GR. We obtain the solution

$$\underline{\alpha}_{GR} = (-kA, kM). \quad (24)$$

The functions A and M give the Schwarzschild solution, and then the constant of motion acquires the standard form

$$\Sigma_0 = \frac{2GM}{c^2}. \quad (25)$$

In other words, in the case of Einstein gravity, the Noether symmetry gives, as a conserved quantity, the Schwarzschild radius or the mass of the gravitating system. This result can be assumed as a consistency check.

In the general case, $f(R) = R^s$, the Lagrangian (17) becomes

$$\mathbf{L} = \frac{sR^{2s-3}[2s + (s-1)MR]}{M} [2(s-1)M^2 A' R' + 2MRM' A' + 4(s-1)AMM' R' + ARM'^2],$$

and the expression (19) for B is

$$B = \frac{s[2(s-1)M^2 A' R' + 2MRM' A' + 4(s-1)AMM' R' + ARM'^2]}{2AMR[2s + (s-1)MR]} \quad (26)$$

As it can be easily checked, GR is recovered for $s = 1$.

Using the constant of motion (24), we solve in term of A and obtain

$$A = R^{\frac{2s^2-3s+1}{s-2}} \left\{ k_1 + \Sigma_0 \int \frac{R^{\frac{4s^2-9s+5}{2-s}} dr}{2ks(s-2)M[2s + (s-1)MR]} \right\} \quad (27)$$

for $s \neq 2$ and k_1 an integration constant. For $s = 2$, one finds

$$A = -\frac{\Sigma_0}{12kr^2(4 + r^2 R)RR'}. \quad (28)$$

These relations allow to find out general solutions for the field equations giving the dependence of the Ricci scalar on the radial coordinate r . For example, a solution is found for

$$s = 5/4, \quad M = r^2, \quad R = 5r^{-2}, \quad (29)$$

⁷ The dimensions are given by R^{1-s} in terms of the Ricci scalar. For the sake of simplicity, we will put $f_0 = 1$ in the forthcoming discussion.

obtaining the spherically symmetric space-time

$$ds^2 = (\alpha + \beta r)dt^2 - \frac{1}{2} \frac{\beta r}{\alpha + \beta r} dr^2 - r^2 d\Omega, \quad (30)$$

where α is a combination of Σ_0 and k and $\beta = k_1$. In principle, the same procedure can be worked out any time Noether symmetries are identified. Our task is now to show how, from a spherically symmetric solution, one can generate an axially symmetric solution adopting the Newman-Janis procedure that works in GR. In general, the approach is not immediately straightforward since, as soon as $f(R) \neq R$, we are dealing with fourth-order field equations which have, in principle, different existence theorems and boundary conditions. However, the existence of the Noether symmetry guarantees the consistency of the chosen $f(R)$ -model with the field equations.

IV. AXIAL SYMMETRY DERIVED FROM SPHERICALLY SYMMETRIC SOLUTIONS

We want to show now how it is possible to obtain an axially symmetric solution starting from a spherically symmetric one adopting the method developed by Newman and Janis in GR. Such an algorithm can be applied to a static spherically symmetric metric considered as a “seed” metric. Let us recast the spherically symmetric metric (4) in the form

$$ds^2 = e^{2\phi(r)} dt^2 - e^{2\lambda(r)} dr^2 - r^2 d\Omega, \quad (31)$$

with $g_{tt}(t, r) = e^{2\phi(r)}$ and $g_{rr}(t, r) = e^{2\lambda(r)}$. Such a form is suitable for the considerations below. Following Newman and Janis, Eq. (31) can be written in the so called Eddington–Finkelstein coordinates (u, r, θ, ϕ) , i.e. the g_{rr} component is eliminated by a change of coordinates and a cross term is introduced [28]. Specifically this is achieved by defining the time coordinate as $dt = du + F(r)dr$ and setting $F(r) = \pm e^{\lambda(r)-\phi(r)}$. Once such a transformation is performed, the metric (31) becomes

$$ds^2 = e^{2\phi(r)} du^2 \pm 2e^{\lambda(r)+\phi(r)} du dr - r^2 d\Omega. \quad (32)$$

The surface $u = \text{costant}$ is a light cone starting from the origin $r = 0$. The metric tensor for the line element (32) in null-coordinates is

$$g^{\mu\nu} = \begin{pmatrix} 0 & \pm e^{-\lambda(r)-\phi(r)} & 0 & 0 \\ \pm e^{-\lambda(r)-\phi(r)} & -e^{-2\lambda(r)} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/(r^2 \sin^2 \theta) \end{pmatrix}. \quad (33)$$

The matrix (33) can be written in terms of a null tetrad as

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \bar{m}^\nu - m^\nu \bar{m}^\mu, \quad (34)$$

where l^μ , n^μ , m^μ and \bar{m}^μ are the vectors satisfying the conditions

$$l_\mu l^\mu = m_\mu m^\mu = n_\mu n^\mu = 0, \quad l_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \quad l_\mu m^\mu = n_\mu \bar{m}^\mu = 0. \quad (35)$$

The bar indicates the complex conjugation. At any point in space, the tetrad can be chosen in the following manner: l^μ is the outward null vector tangent to the cone, n^μ is the inward null vector pointing toward the origin, and m^μ and \bar{m}^μ are the vectors tangent to the two-dimensional sphere defined by constant r and u . For the spacetime (33), the tetrad null vectors can be

$$\begin{cases} l^\mu = \delta_1^\mu \\ n^\mu = -\frac{1}{2} e^{-2\lambda(r)} \delta_1^\mu + e^{-\lambda(r)-\phi(r)} \delta_0^\mu \\ m^\mu = \frac{1}{\sqrt{2}r} (\delta_2^\mu + \frac{i}{\sin \theta} \delta_3^\mu) \\ \bar{m}^\mu = \frac{1}{\sqrt{2}r} (\delta_2^\mu - \frac{i}{\sin \theta} \delta_3^\mu) \end{cases} \quad (36)$$

Now we need to extend the set of coordinates $x^\mu = (u, r, \theta, \phi)$ replacing the real radial coordinate by a complex variable. Then the tetrad null vectors become ⁸

$$\begin{cases} l^\mu = \delta_1^\mu \\ n^\mu = -\frac{1}{2}e^{-2\lambda(r, \bar{r})}\delta_1^\mu + e^{-\lambda(r, \bar{r})-\phi(r, \bar{r})}\delta_0^\mu \\ m^\mu = \frac{1}{\sqrt{2}\bar{r}}(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu) \\ \bar{m}^\mu = \frac{1}{\sqrt{2}r}(\delta_2^\mu - \frac{i}{\sin\theta}\delta_3^\mu) \end{cases} \quad (37)$$

A new metric is obtained by making a complex coordinates transformation

$$x^\mu \rightarrow \tilde{x}^\mu = x^\mu + iy^\mu(x^\sigma), \quad (38)$$

where $y^\mu(x^\sigma)$ are analytic functions of the real coordinates x^σ , and simultaneously let the null tetrad vectors $Z_a^\mu = (l^\mu, n^\mu, m^\mu, \bar{m}^\mu)$, with $a = 1, 2, 3, 4$, undergo the transformation

$$Z_a^\mu \rightarrow \tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma) = Z_a^\rho \frac{\partial \tilde{x}^\mu}{\partial x^\rho}. \quad (39)$$

Obviously, one has to recover the old tetrads and metric as soon as $\tilde{x}^\sigma = \bar{\tilde{x}}^\sigma$. In summary, the effect of the "tilde transformation" (38) is to generate a new metric whose components are (real) functions of complex variables, that is

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} : \tilde{\mathbf{x}} \times \bar{\tilde{\mathbf{x}}} \mapsto \mathbb{R} \quad (40)$$

with

$$\tilde{Z}_a^\mu(\tilde{x}^\sigma, \bar{\tilde{x}}^\sigma)|_{\mathbf{x}=\bar{\tilde{\mathbf{x}}}} = Z_a^\mu(x^\sigma). \quad (41)$$

For our aims, we can make the choice

$$\tilde{x}^\mu = x^\mu + ia(\delta_1^\mu - \delta_0^\mu) \cos\theta \rightarrow \begin{cases} \tilde{u} = u + ia \cos\theta \\ \tilde{r} = r - ia \cos\theta \\ \tilde{\theta} = \theta \\ \tilde{\phi} = \phi \end{cases} \quad (42)$$

where a is constant and the tetrad null vectors (37), if we choose $\tilde{r} = \bar{\tilde{r}}$, become

$$\begin{cases} \tilde{l}^\mu = \delta_1^\mu \\ \tilde{n}^\mu = -\frac{1}{2}e^{-2\lambda(\tilde{r}, \theta)}\delta_1^\mu + e^{-\lambda(\tilde{r}, \theta)-\phi(\tilde{r}, \theta)}\delta_0^\mu \\ \tilde{m}^\mu = \frac{1}{\sqrt{2}(\tilde{r}-ia \cos\theta)} \left[ia(\delta_0^\mu - \delta_1^\mu) \sin\theta + \delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu \right] \\ \bar{\tilde{m}}^\mu = \frac{1}{\sqrt{2}(\tilde{r}+ia \cos\theta)} \left[-ia(\delta_0^\mu - \delta_1^\mu) \sin\theta + \delta_2^\mu - \frac{i}{\sin\theta}\delta_3^\mu \right] \end{cases} \quad (43)$$

⁸ It is worth noticing that a certain arbitrariness is present in the complexification process of the functions λ and ϕ . Obviously, we have to obtain the metric (33) as soon as $r = \bar{r}$.

From the transformed null tetrad vectors, a new metric is recovered using (34). For the null tetrad vectors given by (43) and the transformation given by (42), the new metric, with coordinates $\tilde{x}^\mu = (\tilde{u}, \tilde{r}, \theta, \phi)$, is

$$\tilde{g}^{\mu\nu} = \begin{pmatrix} -\frac{a^2 \sin^2 \theta}{\Sigma^2} & e^{-\lambda(\tilde{r}, \theta) - \phi(\tilde{r}, \theta)} + \frac{a^2 \sin^2 \theta}{\Sigma^2} & 0 & -\frac{a}{\Sigma^2} \\ \cdot & -e^{-2\lambda(\tilde{r}, \theta)} - \frac{a^2 \sin^2 \theta}{\Sigma^2} & 0 & \frac{a}{\Sigma^2} \\ \cdot & \cdot & -\frac{1}{\Sigma^2} & 0 \\ \cdot & \cdot & \cdot & -\frac{1}{\Sigma^2 \sin^2 \theta} \end{pmatrix} \quad (44)$$

where $\Sigma = \sqrt{\tilde{r}^2 + a^2 \cos^2 \theta}$. In the covariant form, the metric (44) is

$$\tilde{g}_{\mu\nu} = \begin{pmatrix} e^{2\phi(\tilde{r}, \theta)} & e^{\lambda(\tilde{r}, \theta) + \phi(\tilde{r}, \theta)} & 0 & ae^{\phi(\tilde{r}, \theta)} [e^{\lambda(\tilde{r}, \theta)} - e^{\phi(\tilde{r}, \theta)}] \sin^2 \theta \\ \cdot & 0 & 0 & -ae^{\phi(\tilde{r}, \theta) + \lambda(\tilde{r}, \theta)} \sin^2 \theta \\ \cdot & \cdot & -\Sigma^2 & 0 \\ \cdot & \cdot & \cdot & -[\Sigma^2 + a^2 \sin^2 \theta e^{\phi(\tilde{r}, \theta)} (2e^{\lambda(\tilde{r}, \theta)} - e^{\phi(\tilde{r}, \theta)})] \sin^2 \theta \end{pmatrix} \quad (45)$$

Since the metric is symmetric, the dots in the matrix are used to indicate $g^{\mu\nu} = g^{\nu\mu}$. The form of this metric gives the general result of the Newman-Janis algorithm starting from any spherically symmetric "seed" metric.

The metric given in Eq. (45) can be simplified by a further gauge transformation so that the only off-diagonal component is $g_{\phi t}$. This procedure makes it easier to compare with the standard Boyer-Lindquist form of the Kerr metric [28] and to interpret physical properties such as the frame dragging. The coordinates \tilde{u} and ϕ can be redefined in such a way that the metric in the new coordinate system has the properties described above. More explicitly, if we define the coordinates in the following way

$$d\tilde{u} = dt + g(\tilde{r})d\tilde{r} \quad \text{and} \quad d\phi = d\phi + h(\tilde{r})d\tilde{r} \quad (46)$$

where

$$\begin{cases} g(\tilde{r}) = -\frac{e^{\lambda(\tilde{r}, \theta)} (\Sigma^2 + a^2 \sin^2 \theta e^{\lambda(\tilde{r}, \theta) + \phi(\tilde{r}, \theta)})}{e^{\phi(\tilde{r}, \theta)} (\Sigma^2 + a^2 \sin^2 \theta e^{2\lambda(\tilde{r}, \theta)})} \\ h(\tilde{r}) = -\frac{ae^{2\lambda(\tilde{r}, \theta)}}{\Sigma^2 + a^2 \sin^2 \theta e^{2\lambda(\tilde{r}, \theta)}} \end{cases} \quad (47)$$

after some algebraic manipulations, one finds that, in $(t, \tilde{r}, \theta, \phi)$ coordinates system, the metric (45) becomes

$$g_{\mu\nu} = \begin{pmatrix} e^{2\phi(\tilde{r}, \theta)} & 0 & 0 & ae^{\phi(\tilde{r}, \theta)} [e^{\lambda(\tilde{r}, \theta)} - e^{\phi(\tilde{r}, \theta)}] \sin^2 \theta \\ \cdot & -\Sigma^2 / (\Sigma^2 e^{-2\lambda(\tilde{r}, \theta)} + a^2 \sin^2 \theta) & 0 & 0 \\ \cdot & \cdot & -\Sigma^2 & 0 \\ \cdot & \cdot & \cdot & -[\Sigma^2 + a^2 \sin^2 \theta e^{\phi(\tilde{r}, \theta)} (2e^{\lambda(\tilde{r}, \theta)} - e^{\phi(\tilde{r}, \theta)})] \sin^2 \theta \end{pmatrix}. \quad (48)$$

This metric represents the complete family of metrics that may be obtained by performing the Newman-Janis algorithm on any static spherically symmetric "seed" metric, written in Boyer-Lindquist type coordinates. The validity of these transformations requires the condition $\Sigma^2 + a^2 \sin^2 \theta e^{2\lambda(\tilde{r}, \theta)} \neq 0$, where $e^{2\lambda(\tilde{r}, \theta)} > 0$. Our task is now to show that such an approach can be used to derive axially symmetric solutions also in $f(R)$ -gravity.

V. AXIALLY SYMMETRIC SOLUTIONS IN $f(R)$ -GRAVITY: AN EXAMPLE

Starting from the above spherically symmetric solution (30), the metric tensor, written in the Eddington-Finkelstein coordinates (u, r, θ, ϕ) of the form (33) is

$$g^{\mu\nu} = \begin{pmatrix} 0 & \sqrt{\frac{2}{\beta r}} & 0 & 0 \\ \cdot & -2 - \frac{2\alpha}{\beta r} & 0 & 0 \\ \cdot & \cdot & -1/r^2 & 0 \\ \cdot & \cdot & \cdot & -1/(r^2 \sin^2 \theta) \end{pmatrix}. \quad (49)$$

The complex tetrad null vectors (37) are now

$$\begin{cases} l^\mu = \delta_1^\mu \\ n^\mu = -\left[1 + \frac{\alpha}{\beta}\left(\frac{1}{r} + \frac{1}{r}\right)\right]\delta_1^\mu + \sqrt{\frac{2}{\beta}}\frac{1}{\sqrt{r}}\delta_0^\mu \\ m^\mu = \frac{1}{\sqrt{2r}}(\delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu) \end{cases} \quad (50)$$

By computing the complex coordinates transformation (42), the tetrad null vectors become

$$\begin{cases} \tilde{l}^\mu = \delta_1^\mu \\ \tilde{n}^\mu = -\left[1 + \frac{\alpha}{\beta}\frac{\text{Re}\{\bar{r}\}}{\Sigma^2}\right]\delta_1^\mu + \sqrt{\frac{2}{\beta}}\frac{1}{\sqrt{\Sigma}}\delta_0^\mu \\ \tilde{m}^\mu = \frac{1}{\sqrt{2}(\bar{r}+ia\cos\theta)}\left[ia(\delta_0^\mu - \delta_1^\mu)\sin\theta + \delta_2^\mu + \frac{i}{\sin\theta}\delta_3^\mu\right] \end{cases} \quad (51)$$

Now by performing the same procedure as in previous section, we derive an axially symmetric metric of the form (48) but starting from the spherically symmetric metric (30), that is

$$g_{\mu\nu} = \begin{pmatrix} \frac{r(\alpha+\beta r)+a^2\beta\cos^2\theta}{\Sigma} & 0 & 0 & \frac{a(-2\alpha r-2\beta\Sigma^2+\sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{2\Sigma} \\ \cdot & -\frac{\beta\Sigma^2}{2\alpha r+\beta(a^2+r^2+\Sigma^2)} & 0 & 0 \\ \cdot & \cdot & -\Sigma^2 & 0 \\ \cdot & \cdot & \cdot & -\left[\Sigma^2 - \frac{a^2(\alpha r+\beta\Sigma^2-\sqrt{2\beta}\Sigma^{3/2})\sin^2\theta}{\Sigma}\right]\sin^2\theta \end{pmatrix}. \quad (52)$$

It is worth noticing that the condition $a = 0$ immediately gives the metric (30). This is nothing else but an example: the method is general and can be extended to any spherically symmetric solution derived in $f(R)$ -gravity.

VI. A PHYSICAL APPLICATION: GEODESICS AND ORBITS

Let us discuss now a physical application of the above result. We will take into account a freely falling particle moving in the space-time described by the metric (52). For our aims, we make explicit use of the Hamiltonian formalism. Given a metric $g_{\mu\nu}$, the motion along the geodesics can be described by the Lagrangian

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \quad (53)$$

where the overdot stands for derivative with respect to an affine parameter λ used to parametrize the curve. The Hamiltonian description is achieved by considering the canonical momenta and the Hamiltonian function

$$p_\mu = \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} = g^{\mu\nu}p_\mu p_\nu, \quad \mathcal{H} = p_\mu\dot{x}^\mu - \mathcal{L}, \quad (54)$$

that results $\mathcal{H} = \frac{1}{2}p_\mu p_\nu g^{\mu\nu}$. The advantage of the Hamiltonian formalism with respect to the Lagrangian one is that the resulting equations of motion do not contain any sign ambiguity coming from turning points in the orbits (see, for example, [29]). The Hamiltonian results explicitly independent of time and it is $\mathcal{H} = -\frac{1}{2}m^2$, where the rest mass m is a constant ($m = 0$ for photons). The geodesic equations are

$$\frac{dx^\mu}{d\lambda} = \frac{\partial\mathcal{H}}{\partial p_\mu} = g^{\mu\nu}p_\nu = p^\mu, \quad (55)$$

$$\frac{dp_\mu}{d\lambda} = -\frac{\partial\mathcal{H}}{\partial x^\mu} = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^\mu}p_\alpha p_\beta = g^{\gamma\beta}\Gamma_{\mu\gamma}^\alpha p_\alpha p_\beta. \quad (56)$$

In addition, since the Hamiltonian is independent of the affine parameter λ , one can directly use the coordinate time as integration parameter. The problem is so reduced to solve six equations of motion. Using the above definitions, it is easy to achieve the reduced Hamiltonian (now linear in the momenta)

$$H = -p_0 = \left[\frac{p_i g^{0i}}{g^{00}} + \left[\left(\frac{p_i g^{0i}}{g^{00}} \right)^2 - \frac{m^2 + p_i p_j g^{ij}}{g^{00}} \right]^{1/2} \right] \quad (57)$$

with the equations of motion

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}, \quad (58)$$

that give the orbits. The method can be applied to the above solution (52) considering the following line element

$$ds^2 = \frac{r(\alpha + \beta r) + a^2 \beta \cos^2 \theta}{\Sigma} dt^2 + 2 \frac{a(-2\alpha r - 2\beta \Sigma^2 + \sqrt{2\beta} \Sigma^{3/2}) \sin^2 \theta}{2\Sigma} dt d\phi + \\ - \frac{\beta \Sigma^2}{2\alpha r + \beta(a^2 + r^2 + \Sigma^2)} dr^2 - \Sigma^2 d\theta^2 - \left[\Sigma^2 - \frac{a^2(\alpha r + \beta \Sigma^2 - \sqrt{2\beta} \Sigma^{3/2}) \sin^2 \theta}{\Sigma} \right] \sin^2 \theta d\phi^2$$

by which the elements of the inverse metric can be easily obtained:

$$g^{tt} = \frac{4\Sigma^2 \left[\Sigma^2 - \frac{a^2 \sin^2 \theta (r\alpha - \sqrt{2}\sqrt{\beta} \Sigma^{3/2} + \beta \Sigma^2)}{\Sigma} \right]}{a^2 \sin^2 \theta (2r\alpha - \sqrt{2}\sqrt{\beta} \Sigma^{3/2} + 2\beta \Sigma^2)^2 + 4\Sigma (a^2 \beta \cos^2 \theta + r(r\beta + \alpha)) \left(\Sigma^2 - \frac{a^2 \sin^2 \theta (r\alpha - \sqrt{2}\sqrt{\beta} \Sigma^{3/2} + \beta \Sigma^2)}{\Sigma} \right)}$$

$$g^{rr} = -\frac{\beta (a^2 + r^2 + \Sigma^2) + 2r\alpha}{\beta \Sigma^2}$$

$$g^{\theta\theta} = -\frac{1}{\Sigma^2}$$

$$g^{t\phi} = \frac{2a\Sigma (-2r\alpha + \sqrt{2}\sqrt{\beta} \Sigma^{3/2} - 2\beta \Sigma^2)}{a^2 \sin^2 \theta (2r\alpha - \sqrt{2}\sqrt{\beta} \Sigma^{3/2} + 2\beta \Sigma^2)^2 + 4\Sigma [a^2 \beta \cos^2 \theta + r(r\beta + \alpha)] \left[\Sigma^2 - \frac{a^2 \sin^2 \theta (r\alpha - \sqrt{2}\sqrt{\beta} \Sigma^{3/2} + \beta \Sigma^2)}{\Sigma} \right]}$$

$$g^{\phi\phi} = -\frac{4\Sigma \csc^2 \theta [a^2 \beta \cos^2 \theta + r(r\beta + \alpha)]}{a^2 \sin^2 \theta (2r\alpha - \sqrt{2}\sqrt{\beta} \Sigma^{3/2} + 2\beta \Sigma^2)^2 + 4\Sigma [a^2 \beta \cos^2 \theta + r(r\beta + \alpha)] \left[\Sigma^2 - \frac{a^2 \sin^2 \theta (r\alpha - \sqrt{2}\sqrt{\beta} \Sigma^{3/2} + \beta \Sigma^2)}{\Sigma} \right]} \quad (59)$$

and the null ones

$$g^{tr} = g^{t\theta} = g^{r\theta} = g^{r\phi} = g^{\theta\phi} = 0. \quad (60)$$

Let us consider the equatorial plane, i.e. $\theta = \frac{\pi}{2}$, $\dot{\theta} = 0$, and assume $\alpha = 1$ and $\beta = 2$. The reduced Hamiltonian can be written as

$$H(r, \theta, \phi, p_r, p_\theta, p_\phi; t) = \frac{2ap_\phi (-2r^3 + r^2 - 1)}{a^2 (-2(r-1)r^2 - 1) + r^5} + \left\{ \left[\left(4a^2 p_\phi^2 (-2r^3 + r^2 - 1)^2 \right. \right. \right. \\ \left. \left. \left. - a^2 (-2(r-1)r^2 - 1) - r^5 \right) (a^2 (r^2 (r(2r-3)(2r+1) + 6) - 2) + (2r+1)r^4) \times \right. \right. \\ \left. \left. \left(-\frac{p_\phi (2r+1)}{a^2 (r^2 (r(2r-3)(2r+1) + 6) - 2) + (2r+1)r^4} - \frac{p_r (a^2 + r^2 + r) + p_\theta}{r^4} - p_r + 1 \right) \right] \right\}^{\frac{1}{2}}. \quad (61)$$

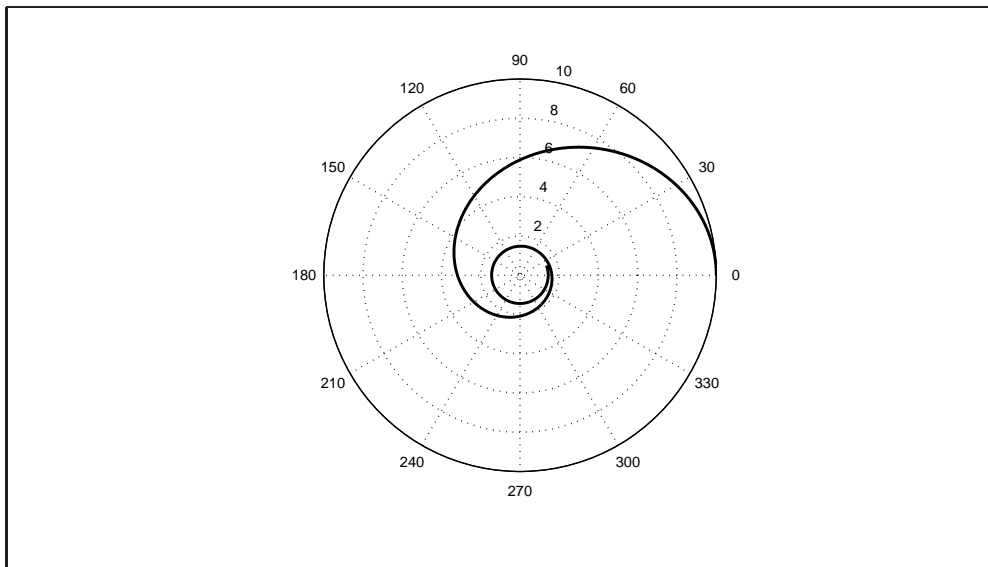


Figure 1: Relative motion of the test particle with $m = 1$.

It is independent of ϕ (i.e. we are considering an azimuthally symmetric spacetime), and then the conjugate momentum p_ϕ is an integral of motion. From Eqs. (58), one can derive the coupled equations for $\{r, \theta, \phi, p_r, p_\theta\}$ and integrate them numerically (the expressions are very cumbersome and will not be reported here). To this goal, we have to specify the initial value of the position-momentum vector in the phase space. A Runge-Kutta method can be used to solve the differential equations. In Fig 1, the relative trajectories are sketched.

VII. DISCUSSION AND CONCLUDING REMARKS

We have shown that the Newman-Janis method, used to derive axially symmetric solutions in GR, works also in $f(R)$ -gravity. In principle, it could be consistently applied any time a spherically symmetric solution is derived. The method does not depend on the field equations but directly works on the solutions that, a posteriori, has to be checked to fulfill the field equations.

The key point of the method is to find out a suitable complex transformation which, from a physical viewpoint, corresponds to the fact that we are reducing the number of independent Killing vectors. From a mathematical viewpoint, it is useful since allows to overcome the problem of a direct search for axially symmetric solutions that, in $f(R)$ -gravity, could be extremely cumbersome due to the fourth-order field equations. However, other generating techniques exist and all of them should be explored in order to completely extend solutions of GR to $f(R)$ -gravity. They can be more general and solid than the Newman-Janis approach. A good source for references and basic features of generating techniques is reference [13]. In particular, the paper by Talbot [14], considering the Newman-Penrose approach to twisting degenerate metrics, provides some theoretical justification for the scope and limitations of adopting the “complex trick”. As reported in Chap. 21 of [13], several techniques can be pursued to achieve axially symmetric solutions which can be particularly useful to deal with non-empty space-times (in particular when perfect fluids are the sources of the field equations) and to deal, in general, with problems related to Einstein-Maxwell field equations. We have to stress that the utility of generating techniques is not simply to obtain a new metric, but a metric of a new spacetime with specific properties as the transformation properties of the energy-momentum tensor and Killing vectors. In its original application, the Newman-Janis procedure transforms an Einstein-Maxwell solution (Reissner-Nordstrom) into another Einstein-Maxwell solution (Kerr-Newman). As a particular case (setting the charge to zero) it is possible to achieve the transformation between two vacuum solutions (Schwarzschild and Kerr). Also in case of $f(R)$ -gravity, new features emerge by adopting such a technique. In particular, it is worth studying how certain features of spherically symmetric metrics, derived in $f(R)$ -gravity, result transformed in the new axially symmetric solutions. For example, considering the $f(R)$ spherically symmetric solution studied here, the Ricci scalar evolves as r^{-2} and then the asymptotic flatness is recovered. Let us consider now the axially symmetric metric achieved by the Newman-Janis method. The parameter $a \neq 0$ indicates that the spherical symmetry ($a = 0$) is broken. Such a parameter can be immediately related to the presence of an axis of symmetry and then to the fact that a Killing vector, related to the angle θ , has been lost. To conclude, we can say that once the vacuum case is

discussed, more general spherical metrics can be transformed in new axially symmetric metrics adopting more general techniques [13]. These approaches will be examined and discussed in future works.

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